Kitaev Model of a 1D topological superconducting wire

Consider Fermions hopping on sites of a 1D wire

$$\mathcal{H}_{hop} = -t \sum_i \left[c_i^{\dagger} c_{i+1} + c_{i+1}^{\dagger} c_i \right]$$

Note : Fermions are <u>spinless</u>

Now I want to add pairing. Since fermions are spinless I cannot add a term like $\Delta c_i^{\dagger} c_i^{\dagger}$ as it will be zero due to Pauli. Therefore I have to add pairing term on neighboring sites.

$$\mathcal{H}_{pair} = \sum_i \left[\Delta c_i^{\dagger} c_{i+1}^{\dagger} + \Delta^* c_{i+1} c_i \right]$$

$\mathcal{H} = \mathcal{H}_{hop} + \mathcal{H}_{pair}$ — is called the Kitaev wire model.

This model has a special point that is very easy to solve:
$\Rightarrow \quad t = \Delta$

WLOG [without loss of Generality] if $\Delta = t$, we can set both to unity. We can always rescale energy if we want a different value.

Kitaev's solution [to think of this way of solving the model you have to be Alexei] using Majorana fermions.

Ettore Majorana did NOT like complex numbers. Hence, he suggested that we should do QM with real numbers only.
For the case of the fermion operator $c_i^{\dagger}$, we can express it using two real operator corresponding to the real and imaginary

⇒ writing a complex (usual) fermion operators using two Majorana operators. That is we take $c_i^\dagger$ and split it into the real and imaginary parts (which I call R_i and L_i)

$$c_i \equiv \frac{1}{2}[R_i - iL_i]$$

$$c_i^\dagger \equiv \frac{1}{2}[R_i + iL_i]$$

real part, imaginary part

⇒ I call these L - for left and R - for right to improve drawing you will see why soon.

⇒ Since R_i and L_i are real, $R_i^\dagger = R_i$, $L_i^\dagger = L_i$

anti-commutation relations:

What is $\{R_i, L_i\}$, $\{R_i, R_i\}$, and $\{L_i, L_i\}$?

⇒ these have to be defined so that anti-commutation relations for $c_i^\dagger$ and c_i's work out correctly

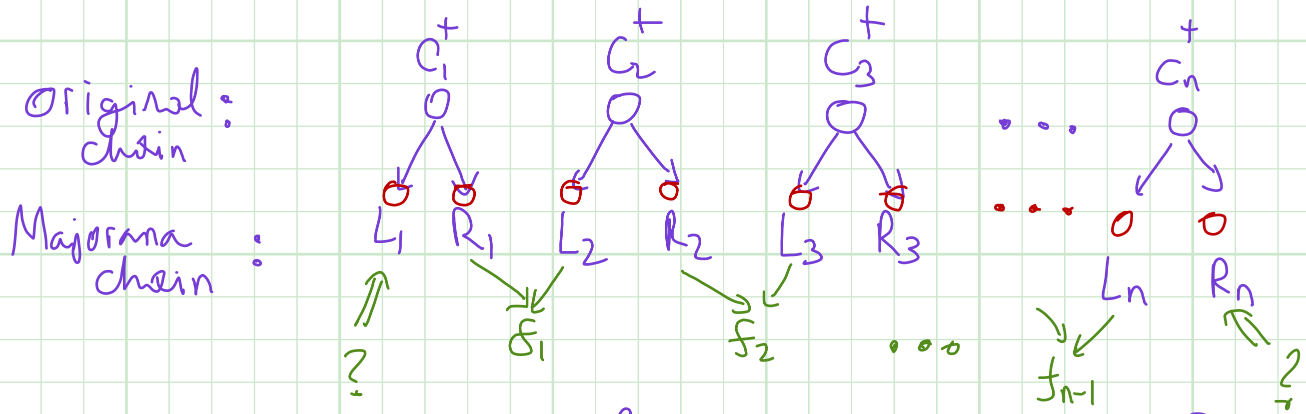
$$1 = \{c_i^\dagger, c_i\} = c_i^\dagger c_i + c_i c_i^\dagger = \frac{1}{4}\left[(R_i + iL_i)(R_i - iL_i) + (R_i - iL_i)(R_i + iL_i)\right]$$

$$= \frac{1}{4}\left[2R_i^2 + 2L_i^2 - \cancel{iR_iL_i} + \cancel{iL_iR_i} + \cancel{iR_iL_i} - \cancel{iL_iR_i}\right] = \frac{1}{2}(R_i^2 + L_i^2)$$

It makes sense that $L_i^2 = R_i^2$ hence we get $R_i^2 = 1 = L_i^2$.

$$0 = \{c_i^\dagger, c_i^\dagger\} = \frac{2}{4}\left[(R_i + iL_i)^2\right] = \frac{1}{2}\left[\cancel{R_i^2} - \cancel{L_i^2} + iR_iL_i + iL_iR_i\right] = \frac{i}{2}(R_iL_i + L_iR_i)$$

Hence we find that $\{R_i, L_i\} = 0$. Now that we know how to use Majorana operators, let's use them to solve the Kitaev Model.



What is H in terms of Majorana operators?

$$H = \sum_i \left[-c_i^\dagger c_{i+1} - c_{i+1}^\dagger c_i + c_i^\dagger c_{i+1}^\dagger + c_{i+1} c_i \right] = \sum_i R_i L_{i+1} \Leftarrow L_1 \text{ and } R_n \text{ not here!}$$

$$\frac{1}{4}\Big(-[R_i + iL_i][R_{i+1} - iL_{i+1}] - [R_{i+1} + iL_{i+1}][R_i - iL_i] + [R_i + iL_i][R_{i+1} + iL_{i+1}] + [R_{i+1} - iL_{i+1}][R_i - iL_i] \Big)$$

$$\Rightarrow -R_iR_{i+1} - R_{i+1}R_i + R_iR_{i+1} + R_{i+1}R_i = 0$$

anti-commute

$$\Rightarrow -L_iL_{i+1} - L_{i+1}L_i - L_iL_{i+1} - L_{i+1}L_i = -2(L_iL_{i+1} + L_{i+1}L_i) = 0$$

$$\Rightarrow -iL_iR_{i+1} + iR_iL_{i+1} - iL_{i+1}R_i + iR_{i+1}L_i + iR_iL_{i+1} + iL_iR_{i+1} - iR_{i+1}L_i - iL_{i+1}R_i$$

$$\Rightarrow -iL_iR_{i+1} - iL_iR_{i+1} + iL_iR_{i+1} + iL_iR_{i+1} = 0$$

$$\Rightarrow iR_iL_{i+1} + iR_iL_{i+1} + iR_iL_{i+1} + iR_iL_{i+1} = 4iR_iL_{i+1}$$

Define $f_i^\dagger = \dfrac{R_i - iL_{i+1}}{2}$

$$f_i^\dagger f_i = \frac{1}{4}(R_i - iL_{i+1})(R_i + iL_{i+1}) = \frac{1}{4}\left[R_i^2 + L_{i+1}^2 + 2iR_iL_{i+1}\right] = \frac{1}{2}\left[1 + iR_iL_{i+1}\right]$$

$$\{R_i, R_i\} = R_iR_i + R_iR_i = 1$$

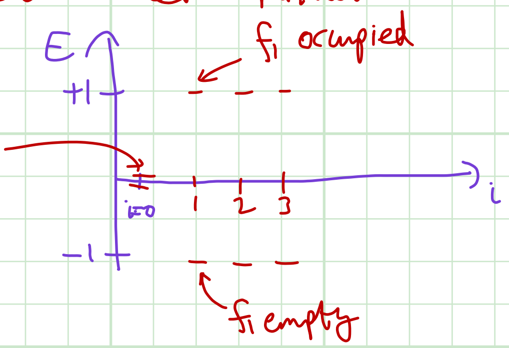
using f operators: $H = \sum_{i=1}^{n-1} (2f_i^\dagger f_i - 1)$

Note: H in terms of L's and R's did not have L_1 nor R_n in it $\Rightarrow$ Energy does not depend on the state of L_1 and R_n $\Rightarrow$ These operators are called Majorana zero modes.

To define the state of L_1 and R_n we must introduce a usual complex fermion $\Rightarrow f_0^\dagger = L_1 + iR_n$

Spectrum of Kitaev model:

Majorana zero mode can be either empty or occupied, Energy of chain is identical.



$\Rightarrow$ MZM good for quantum computation

$\Rightarrow$ Topological protected

$\Rightarrow$ non-locality

$\Rightarrow$ non-abelian statistics

$\Rightarrow$ Could be possible in cold atoms + CM systems

What is topological?

Let us consider the band structure of a modified Kitaev model

$$H = -\sum_i \left(c_i^\dagger c_{i+1} + c_{i+1}^\dagger c_i\right) + \Delta \sum_i \left(c_i^\dagger c_{i+1}^\dagger + c_{i+1} c_i\right) - 2\mu \sum_i c_i^\dagger c_i$$

Strange def will be natural later...

Modifications:

(1) I allow Δ to be finite

(2) I add μ term

Solving the Modified Kitaev model:

This time let us look for a more conventional solution rather than using Majorana fermions. Consider solving an infinite length chain by first Fourier, and second Bogoliubov transforming the Hamiltonian.

FT term by term:

$$\sum_i \Rightarrow k_1 = k_2$$

$$-\sum_i \left(c_i^\dagger c_{i+1} + c_{i+1}^\dagger c_i\right) = -\sum_i \sum_{k_1 k_2} \left(e^{ik_1 r_i - ik_2 r_{i+1}} c_{k_1}^\dagger c_{k_2} + e^{ik_1 r_{i+1} - ik_2 r_i} c_{k_1}^\dagger c_{k_2}\right)$$

use: $r_{i+1} - r_i = 1$

$$= -\sum_{k_1} \left(e^{-ik_1} + e^{ik_1}\right) c^{+}_{k_1} c_{k_1} = -2 \sum_{k} \cos(k)\, c_k^{+} c_k$$

$$\sum_i c_i^{+} c_{i+1}^{+} = \frac{1}{2} \sum_i \left(c_i^{+} c_{i+1}^{+} - c_{i+1}^{+} c_i^{+} \right) = \sum_i \sum_{k_1,k_2} \left(e^{ik_1 r_i + ik_2 r_{i+1}}\, c^{+}_{k_1} c^{+}_{k_2} + e^{ik_1 r_{i+1} + ik_2 r_i}\, c^{+}_{k_1} c^{+}_{k_2} \right)$$

$$= \frac{1}{2} \sum_k \left(e^{-ik}\, c_k^{+} c_{-k}^{+} - e^{ik}\, c_k^{+} c_{-k}^{+} \right) = -i \sum_k \sin(k)\, c_k^{+} c_{-k}^{+}$$

$$\sum_i c_i^{+} c_i = \sum_k c_k^{+} c_k$$

$$H = \sum_k \left[-2\cos(k)\, c_k^{+} c_k - i\sin(k) \left(c_k^{+} c_{-k}^{+} + c_k c_{-k} \right) - \mu\, c_k^{+} c_k \right]$$

As the pairing terms mix fermions with momentum k and $-k$, it is useful to rewrite the Hamiltonian using $k, -k$ pairs:

$$H = \sum_k \begin{pmatrix} c_k^{+} & c_{-k} \end{pmatrix} \begin{pmatrix} \cos(k) + \mu & -i\sin(k) \\ +i\sin(k) & -\cos(k) - \mu \end{pmatrix} \begin{pmatrix} c_k \\ c_{-k}^{+} \end{pmatrix}$$

This Hamiltonian can be solved via a Bogoliubov transform

Let's begin by finding the eigenspectrum in two cases

(1) Kitaev case: $\Delta = 1$, $\mu = 0$

(2) Ordinary insulator case: $\Delta = 0$ $|\mu| > 1$

Kitaev Case:

eigenvalues:

$$\begin{vmatrix} \cos(k) - \varepsilon & -i\sin(k) \\ i\sin(k) & -\cos(k) - \varepsilon \end{vmatrix} = 0 \Rightarrow \varepsilon^2 - \cos^2(k) - \sin^2(k) = 0$$

$$\varepsilon^2 = 1 \Rightarrow \varepsilon = \pm 1 \quad \text{(just as expected)}$$

Insulator Case:

eigenvalues:

$$\begin{vmatrix} \cos(k) - \mu - \varepsilon & \\ & -\cos(k) + \mu - \varepsilon \end{vmatrix} = 0 \Rightarrow \varepsilon = \pm\left(\cos(k) - \mu\right)$$

Note: Both cases correspond to a gapped spectrum:

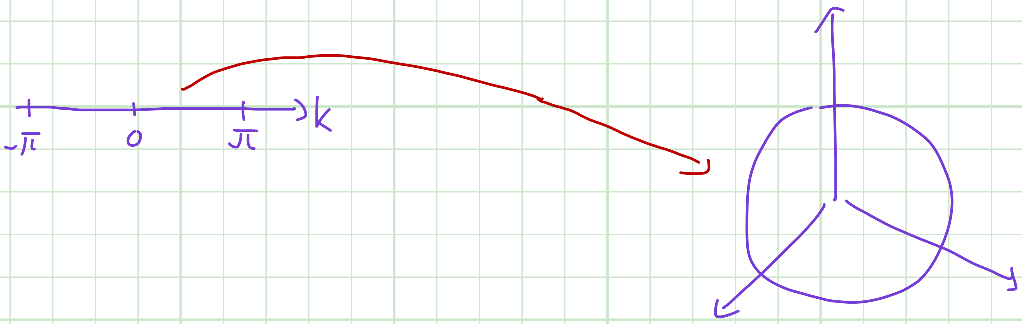
Gapped spectrum is essential for the topological argument.

$\Rightarrow$ let's rewrite the Hamiltonian in the form

$$H = \alpha(k)\,\sigma^x + \beta(k)\,\sigma^y + \gamma(k)\,\sigma^z$$

Since H is everywhere gapped, we can rescale α, β, γ so that $\alpha^2 + \beta^2 + \gamma^2 = 1$

Hence we see that the Hamiltonian is really a function from the BZ to the 3D-sphere of radius 1 (the Bloch sphere).

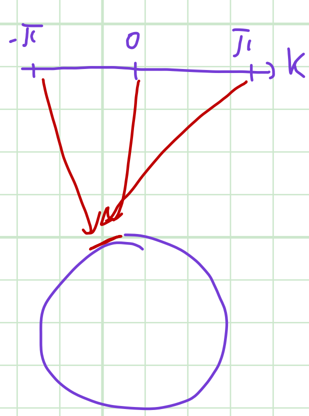


Our Hamiltonian is somewhat special: there is no σ^x, so really we have a mapping from the BZ onto the unit Ring.

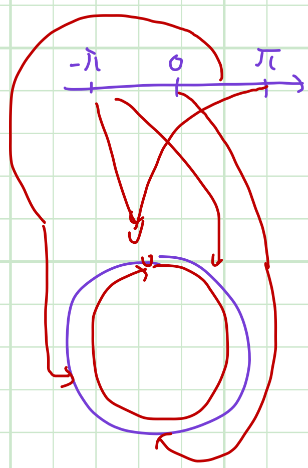
moreover $H(-\pi) = H(\pi)$ so the mapping should be periodic!

What kind of mappings are allowed:

Trivial map (insulator)



Kitaev model



To go from trivial map to non-trivial one we must

(a) close the gap (at some k-point $\Rightarrow$ this let's us add + subtract loops)

(b) deform H into σ^x direction

So as long as (a) + (b) do not occur we have topological protection.

Topology in eigen functions.

Let us now look at the eigen functions of H :

These are the Bogoliubov operators $\gamma_k^+ = u_k c_k + v_k c_{-k}^+$.

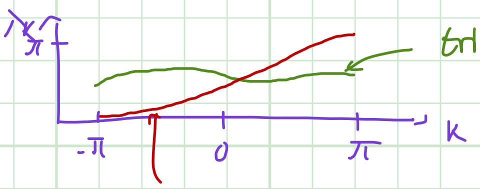
What can we say about u_k and v_k?

(1) Since H is real, we can choose u_k and v_k to be real as well

(2) since $u_k^2 + v_k^2 = 1$, we can introduce an angle λ_k, $u_k = \sin(\lambda_k)$, $v_k = \cos(\lambda_k)$

(3) Eigenvalues are determined up to a phase : if $\{u_k, v_k\}$ is an eigenvalue so is $\{-u_k, -v_k\}$.

Q: what does λ_k do as k goes around the BZ?

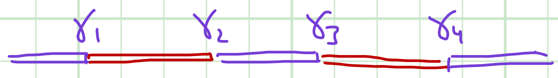


trivial case $\lambda_{-\pi} = \lambda_\pi$ $\Rightarrow$ trivial insulator case

non-trivial case $\lambda_{-\pi} = \lambda_\pi \pm \pi$ $\Rightarrow$ Kitaev phase

What is non-local

(quantum info storage in Majorana fermions)



$f_1^+ = \gamma_1 + i\gamma_2 \qquad f_2^+ = \gamma_3 + i\gamma_4$

→ qubit stored in parity

→ multiple ground states

→ can't affect qubit by local perturbations as it is stored at two ends of the wire.

What is non-Abelian statistics ⇒ next time?